Summary Page:

Our model seeks to determine the best brownie pan to both maximize evenness of heat distribution and baking efficiency in the oven. Although maximizing both simultaneously is impossible, we defined a p-value to assign weights to whether the evenness of heat distribution or baking efficiency would be considered more heavily. In order to calculate the heat distribution, we utilized a professional computational fluid dynamics program to approximate a solution to the heat equation. In order to calculate baking efficiency we began by a theoretical calculation using geometric and tessellation theory for large ovens and then followed with a computer program for smaller ovens. We divided the problem into two cases: those of large, industrial ovens, and those of small ovens such as those that would be found in individual homes. We concluded for industrial ovens that an intermediate conic between a hexagon and a circle would be the optimal shape for a brownie pan, since hexagons have perfect baking efficiency without having too uneven heating, and circles have perfectly even heat distributions. For small ovens, we found that an intermediate conic between a square and a circle would be the most optimal shape for a brownie pan, since squares can tile small ovens most efficiency and again circles provide perfect heat distribution. Our model was able to assign heat distribution and baking efficiency values for the two shapes in consideration for each scenario and their possible combinations, and the best combination of the two shapes in consideration was then calculated using the p-value to create the "Ultimate Brownie Pan".

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Baking Like a Mathematician - The Ultimate Brownie Pan

Team # 18849 · February 4, 2013

1 Problem Introduction

Historically, in the baking of brownies, there has always been much heated debate on how to best bake brownies without sacrificing efficiency. If we want to create brownies as efficiently as possible, we can pack rectangular brownies into the oven to optimize the density of brownie to oven ratio. However, rectangular brownie pans directly harm the quality of the brownie, where the corners provide uneven heating for the brownie. If we want to address this problem by removing the corners, then we will not be able to pack the brownies into the oven as efficiently. Thus, the problem that needs to be addressed is the value of quality and efficiency in baking brownies, and a way to design a brownie pan that best addresses both issues. Our model seeks to calculate the optimal brownie pan that will best bake brownies in respect to both quality and efficiency.

2 Variables/Definitions

1. **Brownie Pan**

   The boundary of each brownie. Each brownie pan will contain an area of A within the boundaries of the brownie pan. The pan is constructed of a material with heat capacity C_p, such that $C_p \ll C_b$ where C_b is the specific heat of the brownie batter.

2. **Brownie Batter**

   The medium in which the heating of the brownie will occur. The brownie batter is a liquid, but because of its high viscosity (empirically determined to be $> 70,000$ centipoise), acts as a solid in the transfer of heat.

3. **Oven**

   The environment in which the brownies are being baked. The environment will be horizontal, and rectangular with side lengths of ratio W/L.

4. **Distribution of Heat**

   The distribution of heat across the brownie crust, a slice of brownie batter a specific distance from the edge of the pan. The value for the evenness of the distribution of heat will be represented as H. Specific ways of calculating H will be discussed in 6.1.

5. **Efficiency of Baking**

   How many brownies one can make using the provided oven space. This is also equivalent to the packing efficiency of a pan into the oven. The efficiency value for a brownie pan will be represented as N.

6. **p-value**

   The amount that we value baking efficiency vs. heat distribution. We define that when $p = 1$, only baking efficiency is considered, and when $p = 0$, only heat distribution is considered, with a full range for intermediate values of p.

3 Assumptions

As one of the primary concerns for this problem is the question of even heat distribution across the brownie, we will make several assumptions that both make our calculations simpler and that most directly calculate the effect of various brownie bans on the heat distribution across the brownie.

1. **The oven is preheated to a constant temperature before brownies are put in.**

   Because of convection, we know that inside the oven, hot air is brought to the top and cooler air remains at the bottom [1]. This implies that there is a difference in temperature across the oven. However, we can ignore this difference with this assumption as across each horizontal layer of the oven, the temperature is constant. Thus, when we put brownies on the same layer, the heat flow into the brownie from the surroundings will not affect the heat flow within the actual brownie. Specifically, we will know that any temperature differentials in the brownie are not results of temperature differentials in the surroundings.

2. **As the brownie is cooked, there is sufficient heat in the environment so that the temperature of the surroundings stays constant.**

   We can make this assumption as we can assume there is a constant heat source coming from the oven to maintain the temperature of the oven. We will use this assumption so we that we will know that the heat flux into the brownie will not be time-dependent, greatly simplifying computational methods.

3. **The density of brownie batter is constant per area within the brownie pan.**

   We will assume that the density of brownie batter is constant over the brownie so that the heat flow through a brownie patch of small area is not changed by the actual brownie batter in that specific area and only by the influx of heat into the specific area. More specifically, the effect that brownie batter has on heat flow is consistent across all brownie patches, regardless of the location and configuration of the patches in the overall brownie.

4. **The direct heating of the brownie by the air is negligible in our calculations.**

   This assumption is due to the fact that our problem addresses the heat differential across the brownie. Because the area of the brownie that is exposed directly to the air is constant, we can assume that the heating as a result of the air will have a constant effect on all parts of the brownie, and therefore not affect the heat differential of the brownie, allowing us to disregard it from our calculations.

5. **Brownie batter is primarily heated through conduction.**

   We know that brownie batter is viscous enough so that there is little convection flow within the brownie fluid [3]. As we have already assumed that the heating from the air is negligible, we can assume that in the heating of the brownie by the pan, heating by conduction will have a much greater impact than heating by convection.

6. **The pan is constant temperature when the brownie begins to be heated.**

   We assume that the pan will reach the equilibrium temperature before the brownie begins to be heated signifcantly as brownie batter has much higher heat capacity than the pan in which the brownie is being baked. Thus, by the time the temperature of the pan equilibrates, there will have been negligible heat transfer between the pan and brownie.

7. **All brownies are cooked for the same amount of time.**

   We make this assumption as we know that the hotter an object gets, the more energy is required to heat it to a hotter temperature. Thus, when considering heat distribution across the brownie, because heat flow is constant into the brownie, the value for the temperature differentials may be affected by the amount of time the brownie has been exposed to heat. By making this assumption, we can make a steady-state approximation.

8. **The amount of heat available is enough to disregard the effects of having multiple brownie pans in the same oven.**

   From our earlier assumption, we have assumed that when brownies absorb heat from the environment, the heat that was absorbed is replaced by the heat source, implying that heat being absorbed by one pan of brownies will not affect the heat being absorbed by another pan. This assumption also greatly simplifies calculations.

9. **The various layers of the oven will be disregarded in the model.**

   We make this assumption on the basis that because of convection, one layer is going to be hotter than the other. However, the amount of heat in the higher layer will be a fixed amount above the heat in the lower layer, and the difference is going to be constant for the whole oven. Thus, regardless of how we design our brownie pans, one layer of the brownies is going to be cooked faster than the other layer, so this fact will be disregarded in our design of the "Ultimate Brownie Pan".

4 Problem Analysis

We first have to consider measures for how even the distribution of heat across the brownie is. We propose the definition of distribution of heat for several reasons. First of all, we know the regardless of the shape of brownie pan, there will be a difference in heat between the outer boundary of the brownie and the center of the brownie. This is due to the heating by conduction entering the pan, pointing inwards, which will affect the outer boundary of the brownie more than the center. Thus, it would not be useful to find the distribution of heat across the whole brownie, as there would be a large range of values that the heat of the brownie could possess. Additionally, we can justify this definition with the existence of isotherms in the conducting of heat in shapes [2]. We know that the distance between consecutive isotherms is not necessarily constant about the center. We also know that the isotherms with highest differential in width will be closest to the irregular boundary, which in this case would be the baking pan. Therefore, when comparing the distance between consecutive isotherms with a width that stays constant about the boundary of the baking pan, if we have increased variation in the distance between isotherms, then we will in turn have a higher variation of concentrated heat. Thus, as we expect the largest heat differentials to be closest to the outer border, by using the thin strip for the calculation of our distribution of heat, we will be able to best model the consistency of the heat throughout the brownie. A formal definition of H is presented in 6.1.

The next part of the problem is the problem of the maximum possible packing efficiency of the brownies into the oven given a specific brownie pan shape. It is important to note that the problem states that the ratio of the oven's sides are of W/L. There are two ways which we could tackle this problem.

1. **Extremely Large Oven - Industrial Baking** - This implies that we can assume a sufficiently large oven such that we only have to take into account the absolute best way of packing the oven, regardless of the boundaries. Suppose, for example, we have a tessellation of hexagonal pans compared to a tessellation of square pans across the oven. If we look specifically at the boundaries, we see that we have empty space in the hexagonal tessellation that does not exist in the square tessellation. However, with sufficiently large oven size, the amount of empty space is negligible, and as both shapes can tessellate a 2D plane completely, both would be ideal solutions for the efficiency of baking.

2. **Small Conventional Oven - Domestic Baking** - This implies that we have some small oven in which a small, finite number of brownie pans can fit. In this case, we would hold A_O constant, and see the effect of varying W/L on the tessellation patterns of the baking pans. It is important to note that the strategies used in the tessellation of a sufficiently large plane will not be the same strategies employed to optimize efficiency in a small plane.

4.1 Method of Attack

We will approach the problem with the following procedure:

1. We have to quantitatively define a value for the quality of the baking based off of the heat distribution across the brownie.

2. We also have to quantitatively define a value for the efficiency of the baking based off of the density of brownies in the oven.

3. We then have to add weights to these values to determine the brownie pan that addresses both issues as best as possible, and relate this to the p value.

This problem asks us not what the best solution for optimal quality or optimal efficiency is, for it seems intuitive that circles would have completely even heating radially, and that square or hexagonal pans would be the best to tessellate across the oven. Rather, the problem wants to know by how far off an unideal solution is, and what weights the derivations from the ideal solution carries in designing the "Ultimate Brownie Pan".

5 Model Design

As mentioned earlier, we have two specific angles at which we have to attack the problem - the heat distribution angle and the baking efficiency angle. As most of the computational calculations required to most accurately model both situations would be too much to do by hand, we will primarily be relying on computer software for our calculations.

5.1 Pan Shapes

We considered various pan shapes, including the most common square- and circle-shaped pans. In addition, we considered the following shapes:

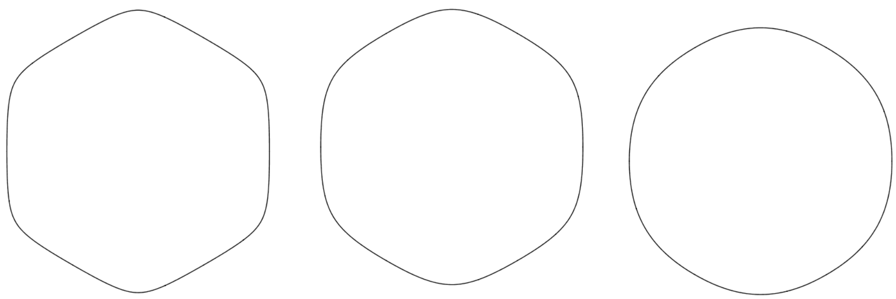
1. Hexagon - Like the square, the hexagon is able to tessalate the plane completely, which means that for large enough ovens, hexagonal pans are essentially as effective at using oven space as the rectangular pans.

2. Conic intermediates between shapes listed above. For example, a conic section can be used to determine a curve intermediate between a hexagon's corner and a circle.

For the second option, we used the following equation to determine the shape of the pan (more specifically, the equation for one-sixth of the pan, which is subsequently rotated to get the entire boundary of the pan):

$$x^2 = 3y^2 - \alpha\left(y - \frac{1}{4}\right)^2 \tag{1}$$

The portion of the curve from $x = -\frac{\sqrt{3}}{4}$ to $x = \frac{\sqrt{3}}{4}$ gives a conic that is tangent to the lines $x = \pm\sqrt{3}y$ at $y = \frac{1}{2}$. We then rotated this portion about $(0, 1)$ by angles of $\frac{k\pi}{3}$ for $k = 0, 1, \ldots, 5$ to get the entire pan shape. For $\alpha = 0$, we get a hexagon, and for $\alpha = 4$, we get a circle. Here are some examples of pan shapes for selected values of α:



These correspond to $\alpha = 0.5, 1, 2.5$, respectively. An analogous set of equations can be used to determine the pan shape for the intermediates between the square and the circle, namely

$$x^2 = y^2 - \beta\left(y - \frac{1}{2}\right)^2 \tag{2}$$

Here β ranges from 0 (square) to 2 (circle). We then take the portion from $x = -0.5$ to $x = 0.5$ and rotate about $(0, 1)$ by $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$ to get the final pan shape.

For all pans analyzed for heat distribution, we based the size off one "standard" pan: all pans analyzed had area equivalent to a 300 mm circle ($\approx 70,685$ mm^2), heights of 50 mm (internal), and had 2 mm thick walls. These are all approximations of common pan sizes for baking, the prototypical "12 inch cake pan" (or, in this case, brownie pan).

5.2 Heat Distribution

We will define the heat distribution formally in 6.1, but in general, we will be looking at the distribution of temperatures in a small section of the crust of the brownie. First, we will simulate the heat flow throughout the brownies in each brownie pan, and will then use a `Python` program to mine the data by implementing complex hulls. Based on the range of temperatures in the small section of crust, we will calculate a value for H that summarizes the important aspects of the distribution of heat in the brownie.

5.3 Baking Efficiency

We will define the efficiency of baking to be directly related to the maximum density of packing of brownie pans into a given oven. This density will be calculated as $\dfrac{KA}{A_O}$ where K is the number of brownie pan that can be put in the oven and A_O is the total area of the oven. We will then take the maximum and minimum possible values for various brownie pan shapes for this density given specific conditions of the oven and scale these values to a range between 0 and 1 in order to best compare the results with the results from the model of the evenness of heat distribution. This is done to best measure the effect of p on both heat distribution and baking efficiency.

We have two different ways to attack the baking efficiency because the packing strategy used for large ovens is different from the packing strategy of small ovens. Sioce the large ovens are easier to consider because as the ovens get larger, the edge effects get smaller, we will be theoretically calculating efficiency values based off of geometric assumptions and implications. However, for small ovens, the edge effect is not negligible, and the packing ability of brownie pans will be heavily reliant on both the shape of the brownie pans and the constraints of the oven. Because this is much more computationally intensive, we will be taking the given assumptions and implications specific to small ovens and calculate the optimal packing using computer software.

6 Heat Distribution Model

6.1 Background

There are many factors involved in the heating and cooking of the brownie, as all three heating methods (conduction, convection, and radiation) are in play. However, convection and radiation are essentially homogeneous over the top of the brownie, as these methods of heat transfer occur across the interface between the brownie and the air. Thus in our model, we consider the disparities in temperature due to the conductive heat transfer between the brownie batter and the side of the aluminum pan.

To determine the heating distribution parameter, H, we will measure how evenly the brownie is being heated at its crust. We set a fixed depth δ and determine the deviation between the maximum and minimum temperatures along all points δ away from the side of the pan.

More precisely, we will set H equal to the following value:

$$H = \frac{\left(\int_C (T_{pan} - T(S))dS\right)^2}{|C|\left(\int_C (T_{pan} - T(S))^2 dS\right)} - 1 \tag{3}$$

Here C is defined as the "crust" of the brownie, or the set of points δ away from the edge of the pan. By the Cauchy-Schwarz inequality, the above quantity will be equal to 0 if and only if the temperature across the crust is uniform, and will be negative otherwise. The magnitude of H gives a good measure of how close the temperature at the crust is to a uniform temperature. For the purpose of computation, it is helpful to give the discrete version of the above equation:

$$H = \frac{\left(\sum_C (T_{pan} - T)\right)^2}{|C|\left(\sum_C (T_{pan} - T)^2\right)} - 1 \tag{4}$$

where $|C|$ now denotes the number of points taken for the crust.

6.1.1 Theoretical Background

In order to calculate the heat distribution throughout the brownie batter in the pan, it is necessary to use the steady-state heat equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{5}$$

In other words, the temperature must vary linearly in x, y and z. The steady state heat equation can be simply derived from Fourier's Law and the conservation of energy. However, solving this equation analytically grows significantly harder for the full three dimensions and for non-uniform objects. Therefore, we use computer programs to make iterative numerical approximations to the heat equation.

6.2 Simulation and Results

In designing our model, we used the following specifications for the brownie batter:

Attribute	Value	Explanation
Conductivity	0.60 W/m·K	This is equivalent to the value for water
Density	1.1 g/cm^3	This value was empirically determined
Specific Heat	4.18 J/g·K	This is equivalent to the value for water

We then used a number of computer programs to solve each case and extract the data, in the process described below:

1. First, we used the `AutoDesk Inventor` program to construct a computer model of the brownie pans, with accurate sizes and materials (aluminum).

2. Next, we imported this model into the `Autodesk Simulation CFD` (Computational Fluid Dynamics) program and assigned brownie batter material properties.

3. After assigning brownie batter material properties, heating conditions were set up on the brownie pan, at a standard baking temperature of 400° F.

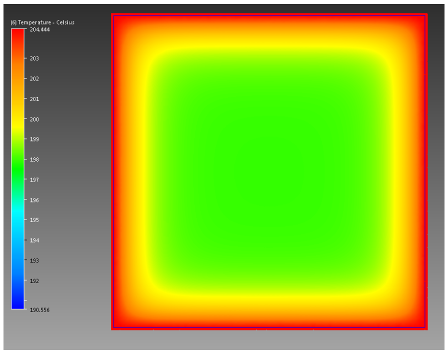
4. We then assigned initial conditions to the brownie batter and brownie pan at a standard 75° F room temperature.

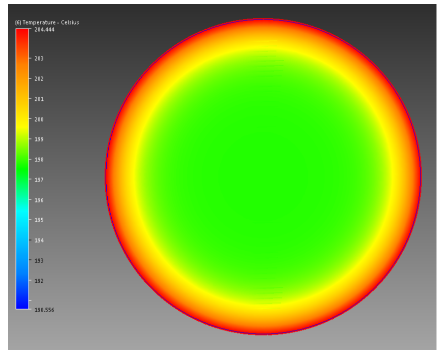
5. Next, we assigned the mesh size for the calculations. Simulation CFD uses numerical methods to approximate solutions to the heat equation through iteration, and therefore must restrict itself to only certain xyz coordinates. The mesh determines which points, or "nodes" in the model it will solve for. Mesh was assigned to provide a balance between computational time and quality of data, so that each model had approximately 100,000-500,000 nodes.

6. Finally, the simulation was run for 250 iterations, which was determined to provide a well-converged result within a reasonable amount of time.

7. Autodesk CFD then automatically outputs the data for each node, and this data was imported into the data-mining program we wrote in `Python`.

The data we generated from the above process can also be visualized. Here are the temperature visualizations for the square and circle cases:





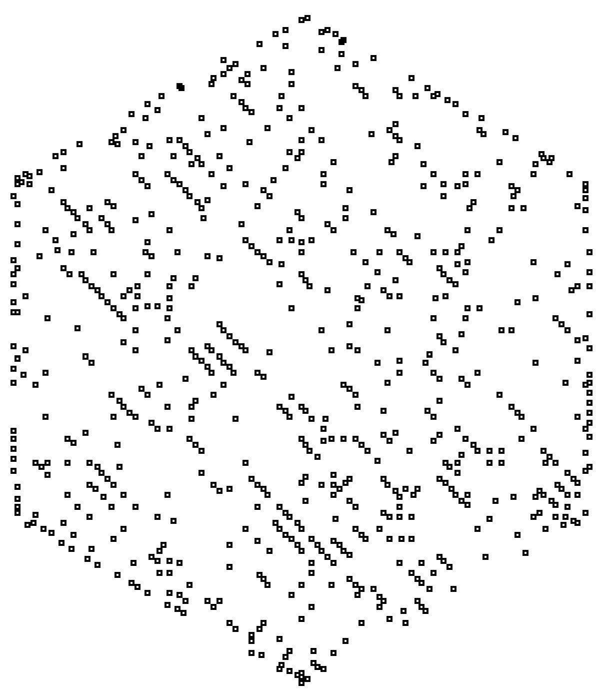
6.3 Data Mining

6.3.1 Explanation of Sorting Code

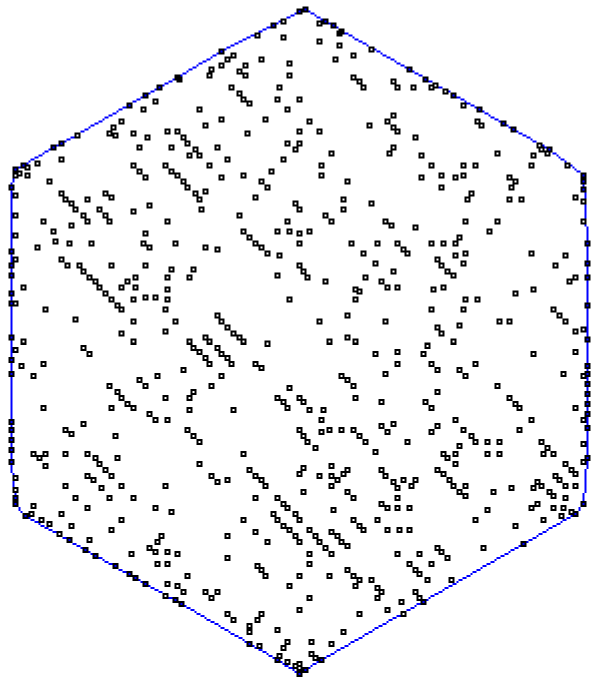
For regular shapes (such as squares and circles), the process of determining the crust is straightforward. In order to determine which nodes lie on the "crust" for irregular shapes (such as the intermediate between the regular hexagon and the circle), we used the following algorithm:

1. Pick a narrow height range $[z_1, z_2]$.
2. Filter the spreadsheet for only nodes whose z-coordinate lies in the interval $[z_1, z_2]$, and project those points onto the xy plane.
3. Determine the convex hull of the projection. (This will be in the shape of the pan).
4. Remove all nodes that are within δ of one of the edges of the convex hull.
5. Determine the convex hull of the remaining points. This will be the "crust".

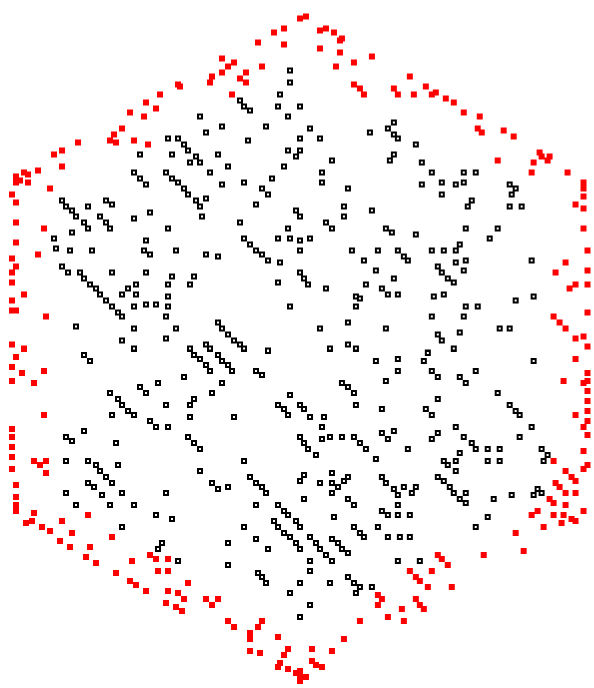
Here is an example of the algorithm at work. Suppose we start with the following set of points (after applying the projection):



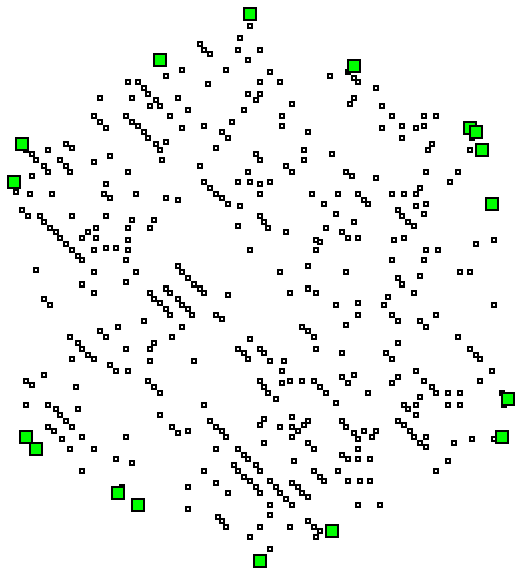
The first step is to draw the convex hull:



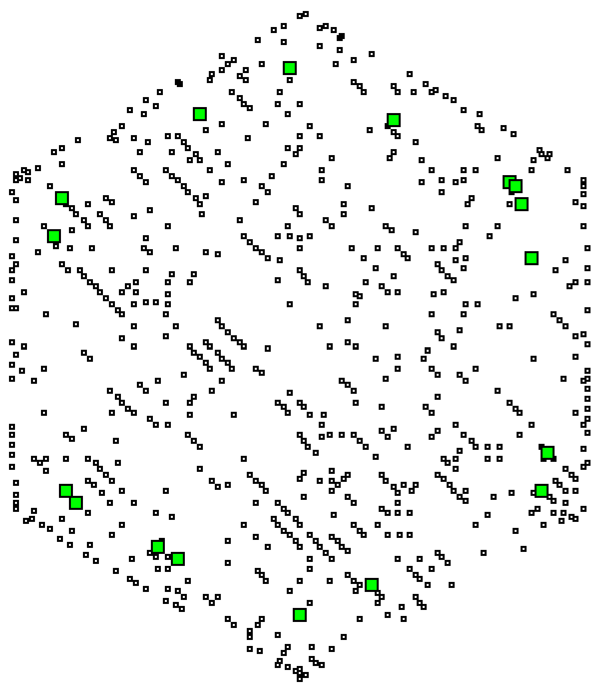
We then remove all points within δ of the edge of the pan. (In this case, $\delta = 20\text{mm}$):



After removing these points, we then mark the convex hull of the remaining points:



Here are the marked points, in the original diagram:



These are the data points that we analyze for our "crust".

We performed this analysis on several different pan shapes.

6.3.2 Results and Evenness of Heating Values

We calculated H for several different shapes, where we wish for a H near zero:

Shape	H ($\times 10^{-5}$)
Circle	-3.96
Hexagon-Circle Conic ($\alpha = 2.5$)	-4.12
Square-Circle Conic ($\beta = 1.5$)	-4.40
Square-Circle Conic ($\beta = 0.5$)	-4.78
Hexagon	-5.38
Square	-5.78

Note that in the above, the square-circle conic approximated a circle less than the hexagon-circle conic approximated a circle, so the results are consistent with what we expect intuitively.

6.4 Strengths and Weaknesses

Strengths

Since our heat distribution model is based on the heat equation, then approximated through iteration to equilibrium, we know that the model will be physically accurate. Furthermore, the physical data for the brownie mixture was empirically determined or estimated based on closely related substances, and values could be specifically measured for more accurate calculations. Our model also looks at the crust of the brownie, and from our background reading, we determined that the greatest variability in isotherms occurs near the edges, particularly corners. Therefore, our model will concentrate measurement on the location that requires measurement most critically. Because our model uses numerical approximations and a convex hull, our model can be applied to any convex shape regardless of symmetry, complexity, or any other factors.

Weaknesses

Hexagon-circle and square-circle hybrids were created on `Autodesk Inventor` by importing a set of points on the surface of the conics then creating a spline curve fit on these rather than well-defined equations for conic sections. However, we used sufficient points (320 per shape) to ensure that our error due to mesh size and computational limitations was greater than our error due to the spline fit. Furthermore, while we were using `Autodesk Inventor 2012`, which does not have equation-graphing capabilities, `Autodesk Inventor 2013` does have equation-graphing capabilities, and could be used to generate exact shapes in the future. Because we used the convex hull on regular polygons, and because regular polygons are convex, the regular polygons utilized significantly less data than was available. However, the convex hull took data points from both the corners and the middle of the edges due to the nonuniform mesh, and thus we were able to mine all of the important data points - those from the edges and those from the corners. This problem does not occur with the circular pan or with any of the conic hybrids.

7 Baking Efficiency Model

We will approach the problem of baking efficiency as essentially a packing problem - we are trying to determine the best way to pack brownie pans of various sizes into different size ovens. It is important to note that in order to maximize baking efficiency in a small, conventional oven, the strategy employed can and will be different than that used to maximize baking efficiency in large ovens.

7.1 Oven of Sufficiently Large Size

The problem originally states that the ovens in which we are baking brownies has side length ratios of W/L. However, since this is a ratio, for sufficiently large ovens, this ratio is unimportant in determining how well we can pack the brownie pans into the oven.

7.1.1 Implications

The construction of our oven as being sufficiently large leads us to a few additional implications that will be key in our model of baking efficiency. The new implications that we make are as follows:

1. **The baking efficiency of a brownie pan is directly proportional to how well it can tessellate a 2D plane.**

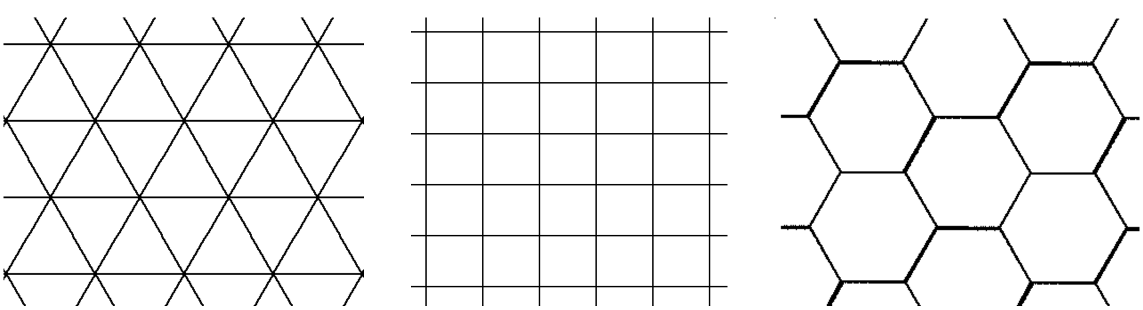
   As ovens get larger and larger, more space is allotted for the tessellation of the oven with the shapes. A direct result of this is that for large ovens, the center of the oven will be filled using the tessellation pattern used to tessellate a 2D plane, leaving a small gap between the tessellated figure and the edge of the rectangular oven. However, as the oven used to bake gets sufficiently large, the ratio of the area of the gap left between the tessellation pattern and the edge of the oven to the area of the tessellation pattern becomes insignificantly small. Thus, for sufficiently large ovens, the baking efficiency is proportional to the density of the figure in a tessellation of a 2D plane.

2. **We will only consider regular polygons in the tessellation of the 2D plane.**

   This implication is on the basis that irregular polygons will inherently have uneven heating distribution. Thus, if we could tessellate an irregular polygon into the 2D plane, there will always be a regular polygon which could tessellate as well as the irregular polygon and have more even heat distribution.

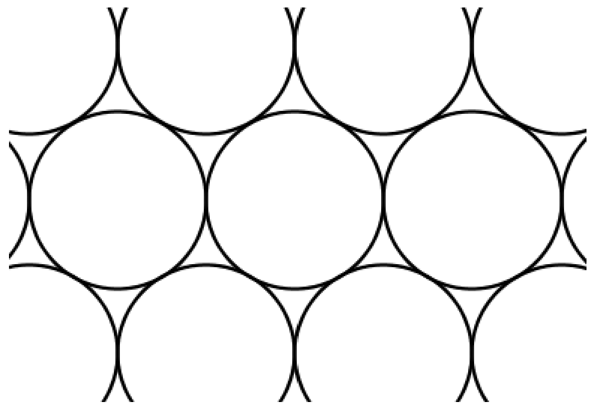
7.1.2 Model and Results

It is a well known theorem that the only regular polygons that can tessellate a 2D plane are the triangle, the square, and the hexagon [4] [5]. In fact, these three tessellations comprise the "fundamental" tessellations of a 2D plane. Under the assumptions that we made, it is clear that, for sufficiently large ovens, the baking efficiency for each of these three pans are the same, and thus no one pan is better than the others in terms of baking efficiency.

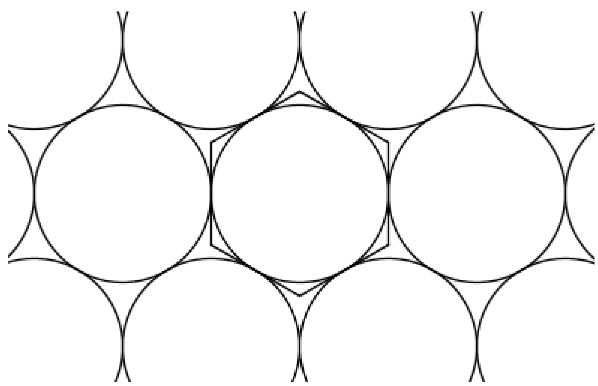


For this case, to determine the shape that we will use to optimize baking efficiency, we must consider the heat distribution for the shapes. It has both already been observed and theoretically determined that the sharper the corners, the more uneven the heat distribution would be. Thus, for sufficiently large ovens, the hexagonal pan will always be the best. When we tessellate a hexagon across a 2D plane, there will be no empty space between hexagons, so the ratio of brownie to oven will approach 1 as edge effects become negligible.

Because circles have the most even heat distribution, we have to consider how we can efficiently fit circles into a 2D plane. We know that the best way to fit congruent circles into a 2D plane is to tessellate it in the same way that we tessellate hexagons. We now need to calculate the density of these congruent circles in 2D space. The circles will be tessellated as follows:



Calculating the density of the circles is as follows. Note how when we take the tangents between each circle, we can use them to construct a regular hexagon about each circle as follows:



We can create a hexagon around each of these circles such that the circles are the incircles of the regular hexagon. The hexagons well then tessellate the plane completely, and the density of the circles in plane can then be calculated as the ratio of the area of the circle to the area of the circle to the area of the hexagon of which the circle is an incircle. Thus, the density of circles in 2D plane is simply $\frac{\pi r^2}{2\sqrt{3}r^2} \approx 0.907$.

As a result, for sufficiently large ovens, we have density values of between 0.907 and 1.000. We can then scale these values appropriately to compare with the evenness of heat distribution in the final creation of the "Ultimate Brownie Pan".

7.1.3 Strengths and Weaknesses

Strengths

Because we can ignore edge effects and concentrate on tessellational efficiency, we can focus on creating a continuous model so that we can vary p continuously. It is also simple to calculate efficiency values for any shape, because we can ignore edge effects and concentrate entirely on the tessellational efficiency of the shape. This value is already known for most polygons, and is already known for all perfectly tessellating polygons and for circles. As a continuation of the previous point, we see that the packing efficiency is a result of the brownie pan and no other factors such as W/L. Finally, because circles tessellate in the same pattern as hexagons, all shapes between a hexagon and a circle will be enclosed in the same hexagonal grid, and so the tessellational efficiency can be easily calculated, despite the complicated nature of these shapes.

Weaknesses

Since we consider the oven to be of infinite size, this model is obviously not completely accurate for real-world situations. However, this model does approximate large ovens, such as those found in industrial settings. Additionally, the data and strategies employed in tessellating across an infinite 2D plane is useful in determining the optimal strategies for tessellation in a small, conventional oven, which will be discussed next.

7.2 Oven of Small, Finite Size

For ovens of small, finite size, we cannot ignore the effect that the ratio W/L has on the packing strategy to optimize baking efficiency. Thus, this becomes a question of figuring out the best way to pack shapes of a certain type, with area A, into an oven of dimensions $W \times L$.

7.2.1 Assumptions

Our model for small, conventional ovens do not use the same assumptions as for the previous model. However, there are a few assumptions that we must make. The new assumptions that we made for modeling these small ovens are as follows:

1. **We will only consider 4 specific ratios of W/L.**

   We make this assumption as the purpose of this subsection of our model is to model the packing efficiency in standard ovens. However, after checking the specifications of the dimensions of various ovens created by many different appliance manufacturers, we found that there are only 4 different sizes of conventional ovens: 30", 27", 24", and 20" [6] [7] [8] [9]. Because we want to make our model as realistic as possible, if these 4 sizes consist of almost all the existing ovens, we only need to consider these 4.

   These 4 ovens have values of W/L are 1.563, 1.375, 1.133, and 1.000 respectively, so we will consider ovens with these values for W/L only.

2. **We will only consider $\frac{A_O}{A} = 2, 3, \ldots, 10$.**

   We feel as though the range between 2 and 10 covers a sufficient variety of sizes for brownie pans. For conventional ovens, for $\frac{A_O}{A} < 2$, the brownie pans would be unrealistically large. Similarly, for $\frac{A_O}{A} > 10$, each brownie pan would only be large enough to make individual brownies, which is also unrealistic. Thus, taking the integer values between 2 and 10 inclusive would give us sufficient data points to draw conclusions on the effect of W/L on the efficiency of packing of brownie pans.

3. **We will only consider square, hexagonal, and circular brownie pans.**

We will consider square and hexagonal pans because they both tessellate a plane perfectly, allowing for the maximum potential packing efficiency, and we must consider circular brownie pans because we know for a fact that they give the most even heat distribution across the brownies. We will not consider triangular brownie pans even though they tessellate a plane perfectly for two reasons. First, in a small finite area of rectangular space, a square will always tessellate better than a triangle, and triangle will only tessellate about as well as a hexagon. Second, triangles have less even heating distributions than squares. For these reasons, for small conventional ovens, triangles are worse than squares in every aspect, so we do not have to consider them. Thus, to get an overview of brownie pans that have the maximal possible efficiency and the most even heating distribution, we only have to consider the 3 aforementioned brownie pans.

7.2.2 Model Design

We need to consider optimal arrangement of squares, hexagons, and circles in a confining rectangle, where edge effects cannot be ignored. To do this, we used a computer program to consider all possible arrangements and arrive at maximal values for each given case.

7.2.3 Model Results

Here are the maximum amount of each brownie pans we could fit in each oven for values of W/L

W/L	A_O/A	Squares	Hexagons	Circles
1.000	2	1	1	1
	3	1	1	1
	4	4	2	2
	5	4	3	3
	6	4	4	4
	7	4	4	4
	8	5	5	5
	9	9	6	5
	10	9	6	6
1.133	2	1	1	1
	3	1	1	1
	4	2	2	2
	5	4	2	3
	6	4	4	4
	7	4	4	4
	8	6	5	5
	9	6	6	5
	10	7	6	6

W/L	A_O/A	Squares	Hexagons	Circles
1.375	2	1	1	1
	3	2	1	1
	4	2	2	1
	5	2	2	2
	6	4	3	3
	7	6	5	5
	8	6	5	5
	9	6	5	6
	10	6	6	6
1.563	2	1	1	1
	3	2	2	1
	4	2	2	2
	5	2	2	2
	6	3	3	3
	7	6	4	4
	8	6	5	6
	9	6	6	6
	10	7	6	6

There are many conclusions we can draw from this data above. First of all, it is important to note that for the above circumstances, a square can fit into the oven as well as or better than a hexagon or a circle in every situation. This draws us to the conclusion that the square will be universally best in baking efficiency if we are limited to a small, conventional oven. It is important to note that this is not completely true; for example, it has been determined that, given side lengths of a square as s, if the side lengths of the oven are $1.99s$ then you can pack one square in the oven, but 2 hexagons/circles. However, the number of times that this occurs is insignificantly small, and as a general trend, squares will pack better than hexagons and circles.

Secondly, it is important to note that, in every case above, hexagons and circles will have approximately the same packing efficiencies. Because circles have a more even heating distribution than hexagons and hexagons do not present much of an advantage as compared to circles in terms of packing, we do not need to consider hexagons in the packing of small, conventional ovens.

The fact that we only have data at specific points rather than a continuous line of data makes it a bit more difficult to calculate a good metric for efficiency, as the values of density varies with various oven sizes, and there is no continuous expression that would be able to easily determine the density of the brownies as oven size changes.

7.2.4 Strengths and Weaknesses

Strengths

One of the main strengths of this model is that it is realistic in that it follows the parameters as given by traditional ovens and approximates brownie pan sizes based off of those oven sizes, which makes this model applicable. Additionally, it covers a wide variety of brownie pan sizes in the varied values of A_O/A. From our model, we were able to draw a firm conclusion about the efficiency of packing square pans as compared to circular and hexagonal pans, and have shown that our conclusion applies for almost all sizes of both ovens and pans.

Weaknesses

The main weakness of our model is that there is no continuous definition of the efficiency as A_O/A moves across real numbers instead of just integers. There are several implications to this. First of all, as a result of the effect of edge limitations, the difference in baking efficiency of square pans and circular pans cannot be explicitly determined from the value of A_O/A and W/L, which makes it hard for us to assign efficiency values. Additionally, because of our methodology in testing our model, it makes it hard to determine the efficiency of packing of shapes that are between square and circular pans. However, this main weakness can be resolved by graphing the obtained data points for best possible packing and using the fact that we know that the efficiency of the in between shapes can be bound between the two graphs to help us determine efficiency values that can be used for comparison.

7.3 Summary of Results

From the data obtained from our two models, we have seen that there are two different strategies that must be employed to maximize baking efficiency for various sizes. If we want to optimize baking efficiency in a sufficiently large industrial-sized oven, we should use hexagonal pans since they tessellate as well as squares but have better heating distribution. If we want to optimize baking efficiency in a small, conventional oven, then we should use square pans to pack into the oven because of the edge effect.

However, in both cases, we have determined the effect that the shape of brownie pan on the packing, giving a well-defined explanation of how the hexagonal and square pans are more efficient than the circular pan. Using this data, we can assign efficiency values to the shapes in both cases and use this information, along with p, to determine the optimal value for H.

8 Combining the Two Models

In this section, we will combine the models for heat distribution and baking efficiency. Since we have separate models for baking efficiency for small and large ovens, we will consider each separately. However, we will approach the comparison between the two models in a similar fashion.

The first thing we need to do in order to compare the heat distribution value and the efficiency value for the model is to scale them both down. By putting them on equal scales, we make it such that the only factor that affects the comparison of the two values is the p-value. In order for us to compare the values appropriately, we will scale both values to a range from 0 to 1. We will make 1 the best possible value and 0 for the worst considered possible value. Thus, for the heating distribution, we will assign the circle a value of 1 and the hexagon or square a value of 0. Similarly, for the baking efficiency, we will assign the circle a value of 0 and the hexagon or square a value of 1.

We will then attempt to create a continuous graph between the two extreme values for both the heat distribution and efficiency values as the shapes transform from a circle to either a square or hexagon. As a result, we will be able to have values for both heat distribution and efficiency for all shapes between a circle and either a square or hexagon, and we will be able to find the optimize shape for various values of p.

In the large oven model, we have determined that a hexagon is the best shape to tessellate the oven. For the combination of the two models, we will then consider the hexagon and the circle and all shapes in between to calculate the optimal shape for the “Ultimate Brownie Pan”. Similarly, because we have determined that a square is the best shape to tessellate the oven for small ovens, to create the "Ultimate Brownie Pan", we will need to calculate the optimal conic shape between a circle and a square.

8.1 Heat Distribution and Efficiency Values

We will approach the standardization of the heat distribution (H) and baking efficiency (N) values for both models in a similar manner.

- **Heat Distribution Value**

  For the calculation of the heat distribution value, we will apply a regressional fit to the data points we have for values of H as they relate to α (namely 0, 2.5, and 4), as well as β (0, 0.5, 1.5, and 2).

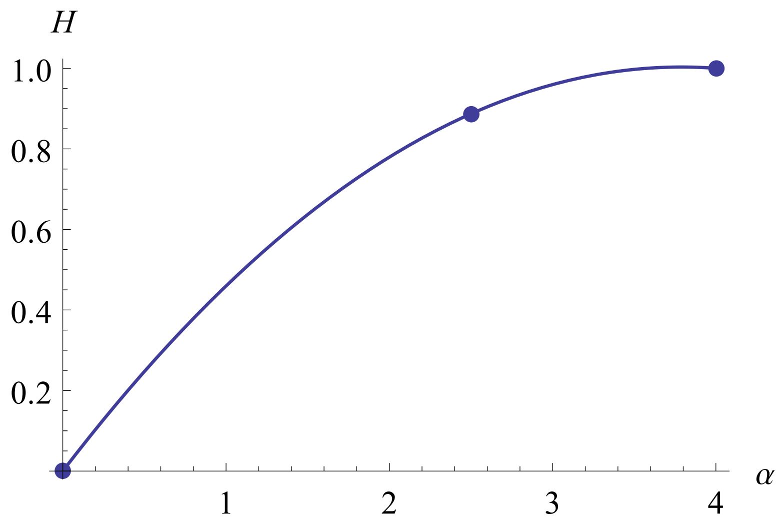
- **Efficiency Value**

  For the efficiency value, we will use calculus to derive an explicit formula for the density of the area based on α. Because we know that the equation of the conic section depends on the value of α, we can calculate a formula for the area of the conic section. We will then get a continuous function for the area of the shape as alpha ranges from the minimum to the maximum. We know that the N will be able to be directly related to this value as for packing, we consider the area of the conic divided by the best area for tessellation (hexagon or square). We will then use a linear transformation to scale the range of points to 0 to 1 for comparison with H.

  We will apply the same method to model the small conventional oven even though how well a conic section fits does not follow in the same exact manner as in the large oven for the following reasons. First of all, the efficiency value for a brownie pan should be the best tessellation of a shape as compared to the best and worst shapes for tessellating the oven, and we have shown in our model that the square shape sets an upper bound and the circle sets a lower bound for packing into the small oven. Additionally, because of the edge effect, there will be no clear way to define efficiency for the small manner as a direct relationship to the density because the density is not well defined for all values of A_O/A, so there is no better way to determine efficiency. Finally, we are justified in using this method as we are referencing the large oven and we know that this method does work for other models, which makes this the optimal method for calculating efficiency of the small oven model.

8.1.1 Large Oven Model

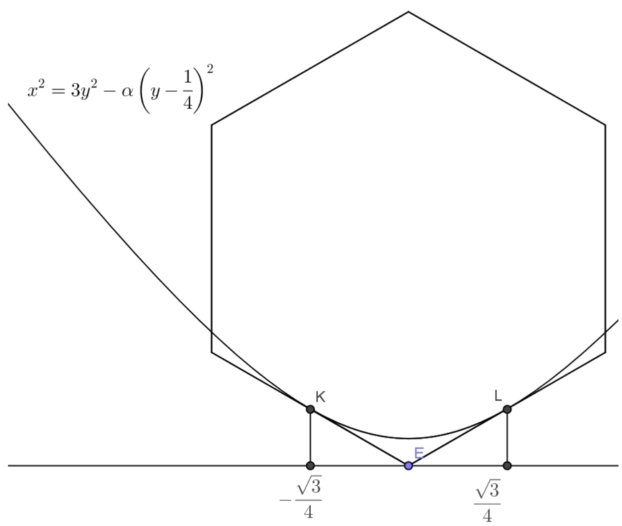
For the large oven model, we will begin by creating a relationship between H and α. Thus, we plotted our H values against α.



The values for H can be approximated by a quadratic fit, and are given by:

$$H = -0.06987\alpha^2 + 0.5295\alpha \tag{6}$$

We now need a relationship between the baking efficiency (N) and α. We first relate the packing efficiency (E) to α. Because packing efficiency is independent of the area of our pan (only on the shape), we will assume without loss of generality that the side length of the hexagon is 1.



We then wish to calculate the "wasted" area. Notice that curved triangle KEL is exactly one-sixth of the total wasted area. To get the area of KEL, we first need to solve for y in terms of x. From equation 1, we have $x^2 = 3y^2 - \alpha\left(y - \frac{1}{4}\right)^2$. Expanding and solving for y gives

$$y = \frac{-\frac{\alpha}{4} + \sqrt{\left(\frac{\alpha}{4}\right)^2 + (3-\alpha)\left(\frac{\alpha}{16} + x^2\right)}}{3-\alpha}$$

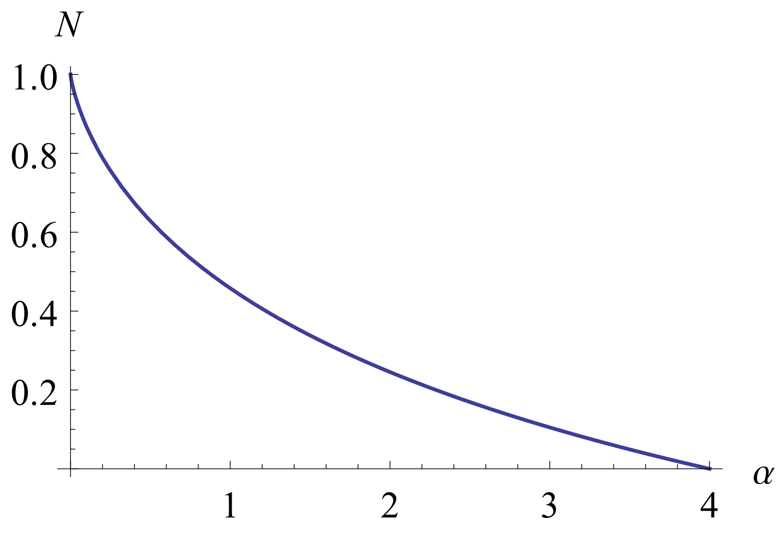
Thus our total wasted area can be given by the following expression:

$$A_{wasted} = (1 - E(\alpha))A_{total} = 6\left[\int_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}} \frac{-\frac{\alpha}{4} + \sqrt{\left(\frac{\alpha}{4}\right)^2 + (3-\alpha)\left(\frac{\alpha}{16} + x^2\right)}}{3-\alpha} d\alpha - \frac{\sqrt{3}}{16}\right] \tag{7}$$

Thus using the fact that $A_{total} = \frac{3\sqrt{3}}{2}$, we can solve for $E(\alpha)$ by evaluating the above expression:

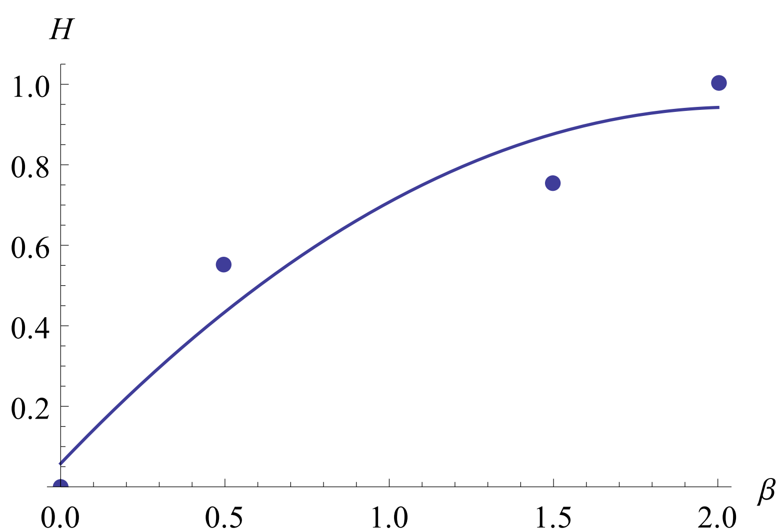
$$E(\alpha) = \begin{cases} 1 - \frac{a}{4(\alpha-3)}\left(1 - \sqrt{\frac{3}{3-\alpha}}\sinh^{-1}\left(\sqrt{\frac{3}{\alpha}-1}\right)\right) & 0 < \alpha < 3 \\ \frac{11}{12} & \alpha = 3 \\ 1 - \frac{a}{4(\alpha-3)}\left(1 - \sqrt{\frac{3}{4(3-\alpha)}}\tan^{-1}\left(\frac{\sqrt{12(\alpha-3)}}{6-\alpha}\right)\right) & 3 < \alpha < 4 \end{cases} \tag{8}$$

To get our normalized value of N, we apply the linear transformation $N(\alpha) = \frac{E(\alpha)-E(4)}{E(0)-E(4)}$, so that $N(0) = 1$ and $N(4) = 0$. Here is the plot of N versus α:



8.1.2 Small Oven Model

For the small oven model, we will approach the combination of the two values in the same manner as the large oven, as previously mentioned. However, since we found when analyzing the baking efficiency in 7.2 that a square will tile small ovens better than a hexagon, we choose to use a square for small ovens. To create a relationship between H and β, we use our H values plotted against β:



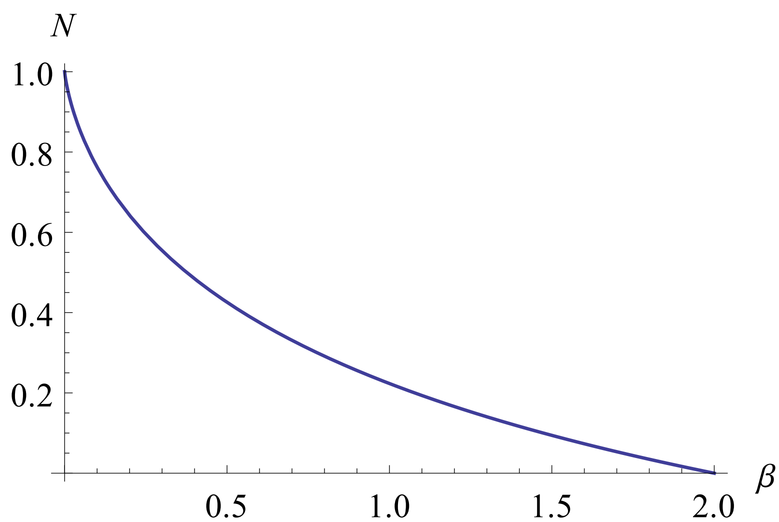
The values for H can be approximated by a quadratic fit, and are given by:

$$H = -0.2073\beta^2 + 0.8569\beta + 0.0578 \tag{9}$$

To determine a relationship between N and β, we proceed very similarly as in the hexagonal case. We arrived at the following formula for $E(\beta)$:

$$E(\beta) = \begin{cases} 1 - \frac{\beta}{\beta-1}\left(1 - \frac{\sinh^{-1}\left(\sqrt{\frac{1}{\beta}-1}\right)}{\sqrt{1-\beta}}\right) & 0 < \beta < 1 \\ \frac{5}{6} & \beta = 1 \\ 1 - \frac{\beta}{\beta-1}\left(1 - \frac{\tan^{-1}(\sqrt{\beta-1})}{\sqrt{\beta-1}}\right) & 1 < \beta < 2 \end{cases} \tag{10}$$

Again, we apply the linear transformation of E to get N, using the formula $N(\beta) = \frac{E(\beta)-E(2)}{E(0)-E(2)}$. Graphing this relationship, we get a graph for N as it relates to β as follows:



Now we have created relationships between H and β and N and β for both models.

8.2 Results/Relationship between p and Nature of Conic Sections

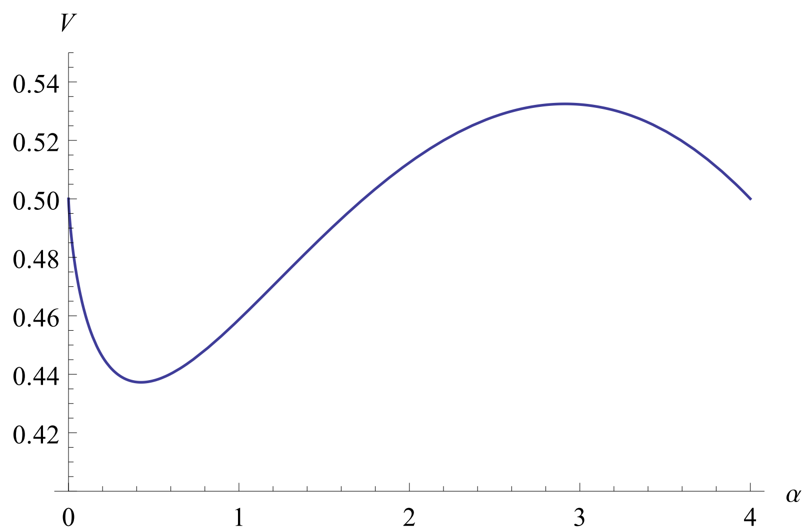
To determine the best pan for a given p value for both models, we simply have to find the maximum value of the following equation

$$V = pN + (1-p)H \tag{11}$$

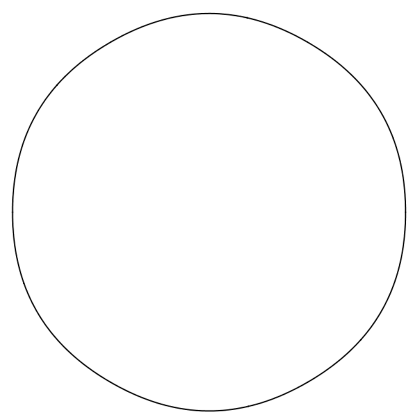
Where V is the total value of the brownie pan for a given p-value. Again, we will divide our consideration into large ovens (those with hexagonal tiling) and small ovens (those with square tilings).

8.2.1 V for Large Ovens

We will consider a range of values for p between 0 and 1. Let us test, for example, when we take into consideration heat distribution and baking efficiency equally. Thus, the p-value for our given parameter would be $p = 0.5$. When we plug in $p = 0.5$ into our equation for V, we get the following graph.



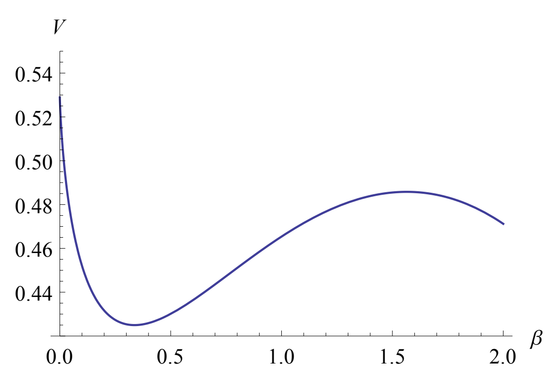
For a p of 0.5, we maximize our V by using an α of 2.914, which would result in a pan that looks as follows.



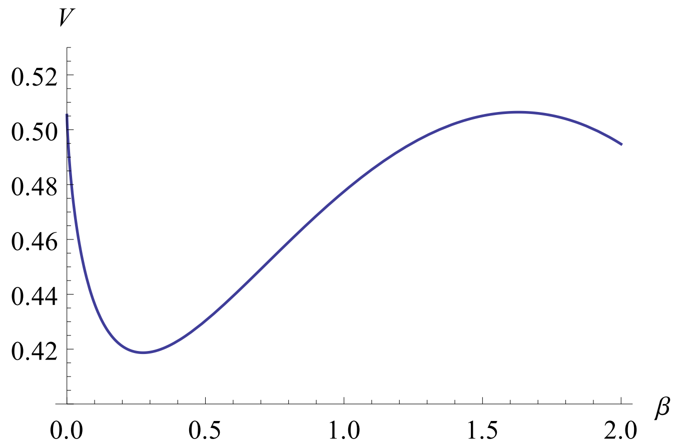
As you can notice, it very much resembles a circle. However, when $p > 0.517817$, the optimal solution is always a hexagon. When $p < 0.517817$, the optimal α value is more than 2.83, which is very similar to a circle (but is never a perfect square).

8.2.2 V for Small Ovens

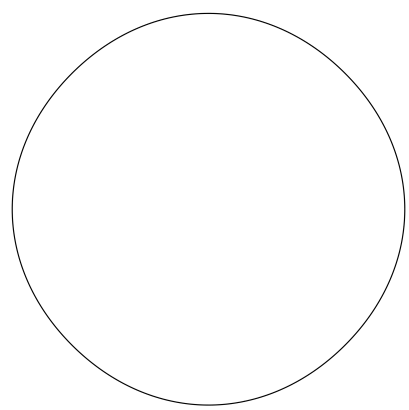
Whereas the optimal shape for hexagonal shapes was very similar to a circle, the optimal shape for $p = 0.5$ for small ovens turns out to be a square. Here is the plot of V versus β for $p = 0.5$:



In this case, V is maaximized at $\beta = 0$, meaning that our combination of H and N is maximized when our pan shape is a square. In fact, whenever $p > 0.475$, the optimal solution is always a square. However, at $\beta = 0.475$, the maximum is attained at two places:



One maximum is attained at $\beta = 0$, which represents a square. The other maximum is at $\beta = 1.629$, which corresponds to the following shape:



Which is again very close to a circle. As with the large oven case, for $p < 0.475$, the optimal shape will never reach a circle, but will come very close.

In both cases, it is interesting to notice that there was a threshold T for which for $p > T$, the preferred pan shape was always the polygonal shape (the square or the hexagon, but for $p < T$, the optimal shape will be a shape closer to a circle (but only a circle when $p = 0$).

8.3 Strengths/Weaknesses

Beyond the strengths and weaknesses discussed in the individual models, the following strengths and weaknesses arise through the combination of the Heat Distribution and Baking Efficiency models:

Strengths

The continuous regression and calculations of both H and N allow us to calculate the best pan shape (α value) for all p values, for the situation of both large ovens and small ovens. One other strength is that we do look at the values separately for large and small ovens, since they are clearly distinct cases.

Weaknessess

The fit for the heat distribution had to be approximated on only a few points of data, and a well-defined curve was not able to be generated. With more time, we would be able to add points to increase the accuracy of our approximation. Furthermore, our packing data for small ovens is incomplete for conic intermediates between squares and circles, though we do know that squares represent an upper bound and circles represent a lower bound.

9 Conclusions

We have created a way to determine the best possible brownie pan (taking into consideration evenness of heat distribution and baking efficiency). First we examined the evenness of heat distribution based on the brownie pan shape. We did this by approximating the solution to the heat equation for each pan shape using `Autodesk CFD` software. We then determined the baking efficiency of various shapes in two ways. We determined the maximum baking efficiency for large ovens theoretically, and the maximum baking efficiency for small ovens computationally. We combined the two models by performing a linear transformation on calculated values for heat distribution and baking efficiency to make them useful for analysis. We then finished the analysis by creating a continuous function describing the heat distribution and baking efficiency values for all shapes. Thus, when we incorporate a weight p for how much we value heat distribution over baking efficiency and vice versa, we will have a well-defined function for calculating the optimal shape of the "Ultimate Baking Pans".

10 Software Used

- `Autodesk Inventor 2012`: We used this program to design the brownie pans for use in the heat distribution calculations.
- `Autodesk CFD 2013`: We used this program to approximate the heat equation for each brownie pan shape computationally.
- `OriginPro 8`: We used this program to perform regressions on the data for H values.
- `Mathematica 9`: We used this program to solve some equations and to generate all of our graphs.
- `GeoGebra 4`: We used this program to generate figures and points for the conic intermediates between hexagons and circles and between squares and circles.
- `Python 2.7`: We used this to write our data-mining program.

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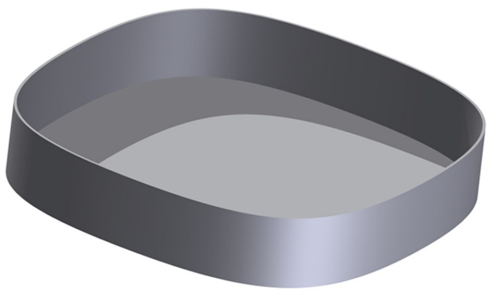
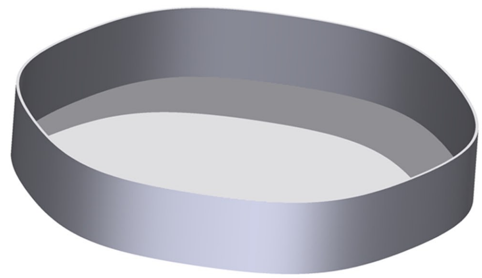
Ultimate Brownie Pan





Households

Better packing efficiency for small ovens

Industrial

Better packing efficiency for large ovens

...because uneven heating is for squares